INDIANA UNIVERSITY

BORKENSTEIN COURSE

Center for Studies of Law in Action

Practice Mathematical Problems Relevant to Breath and Blood Alcohol Testing Programs

by
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Practice Problems

(Use the attached standard normal, t-tables and equations)

1. An individual, male aged 35 weighing 180 lbs., begins drinking at 7 pm and stops at 11 pm. He is arrested for DUI at 1 am the next morning and is administered a breath test at 2 am. The duplicate breath alcohol results are 0.135 and 0.143 g/210L. Determine the estimated number of 12 fluid ounce beers (assume 4% alcohol by volume) that the individual would have consumed along with a ±25% interval of uncertainty. Assume Widmark's $\rho = 0.70$ and $\beta = 0.017$ g/100ml/hr. Use Equation 1.

2. An individual, female aged 30 and weighing 125 lbs., begins drinking at 9 pm and continues until 12 midnight. She consumes eight drinks, each containing one fluid ounce of 80 proof vodka. Estimate what her blood alcohol concentration would be at 2 am the next morning along with a ±40% interval of uncertainty. Assume Widmark's $\rho=0.60$ and $\beta = 0.015$ g/100ml/hr. Use Equation 1.

3. An individual, male aged 25 weighing 160 lbs, begins drinking at 8pm and stops at 1 am. During this time he consumes nine 12 fluid ounce beers (assume 4% by volume) and five glasses of vodka each containing one fluid ounce of 80 proof. Estimate what his blood alcohol concentration would be at 2 am along with a ±25% interval of uncertainty. Assume Widmark's $\rho=0.72$ and $\beta = 0.018$ g/100ml/hr. Use Equation 1.

4. An individual, male aged 45 weighing 190 lbs., begins drinking at 6 pm and stops at 10 pm. He is arrested for DUI at midnight and is administered a breath test at 1 am. The duplicate breath alcohol results are 0.092 and 0.085 g/210L. Determine the estimated number of 12 fluid ounce beers (assume 4% alcohol by volume) that the individual would have consumed along with a ±25% interval of uncertainty. Compute also the 2 standard deviation uncertainty interval using Widmark's uncertainty equation. Assume Widmark's $\rho = 0.70$ and $\beta = 0.014$ g/100ml/hr. Use Equations 1 and 2.

5. An individual, male aged 38, weighing 170 lbs and height of 5 feet 10 inches, begins drinking at 7pm and stops at 11pm. He is arrested for DUI at midnight and is administered a breath test at 1am with results of 0.145 and 0.152 g/210L. Determine the estimated number of 12 fluid ounce beers (assume 4% alcohol by volume) that he consumed using both Widmark's equation and the equation of Watson, et.al., for total body water (TBW). Assume $\rho = 0.72$ and $\beta = 0.018$ g/100ml/hr.

Determine also a ±25% interval of uncertainty. Use equations 1, 15 and 16.

6. Assume that an individual provides duplicate breath samples resulting in 0.097 and 0.106 g/210L. When a new simulator solution (reference value = 0.083 g/210L) was installed on the instrument about one month earlier the first three results were: 0.081, 0.083 and 0.080 g/210L. Correct the mean BrAC for any bias and determine the 99% confidence interval for the within-subject population mean breath alcohol concentration. Find the standard deviation from: SD = 0.0305B+0.0026 and use t=2.57. Use Equations 3 and 4.

7. An individual provides two breath samples that result in 0.084 and 0.087 g/210L. First determine and draw the 99% confidence interval using t=2.57. Next determine the probability that the individual's true mean breath alcohol concentration exceeds 0.080 g/210L. Assume that there is no bias and that the combined uncertainty for the mean is 0.0032 g/210L. Use Equations 3 and 6.

8. Assume that you have performed duplicate analyses on a blood sample by headspace gas chromatography and obtained 0.088 and 0.089 g/100ml. The control standards used during the same run as the subject's samples were purchased from Cerilliant who provided a certificate of analysis stating the reference value was 0.100 g/100ml with a combined uncertainty of 0.0003 g/100ml. There were n=8 measurements of this control during the run with a mean result of 0.1025 g/100ml and a standard deviation of 0.0011 g/100ml. The maximum bias observed for these eight control measurements was 0.002 g/100ml and no bias correction was made. An automatic dilutor was used which had a certificate of analysis stating that based on n=10 gravimetric measurements the reference volume delivered would be 10.12 µL with an uncertainty estimate of 0.05 µL. Finally, a total method uncertainty estimate was available from the analysis of thousands of duplicate results yielding the equation:
u_{method} = 0.0105C + 0.0004 Determine the combined uncertainty along with a 95% confidence interval for the mean of the subject test results. Use t = 1.96. Also, develop a table showing the percent contribution from each component to total uncertainty.

9. An individual provides duplicate breath samples resulting in 0.085 and 0.098 g/210L. Compute the 95% confidence interval for the difference between these results and determine if this difference is acceptable. Using the same form of analysis determine the acceptable difference that should be allowed at the 95% level for mean BrAC results of 0.250 g/210L. Assume the combined uncertainty of BrAC measurement is determined from SD = 0.0305B+0.0026.

10. An individual provides duplicate breath samples resulting in 0.088 and 0.091 g/210L in an instrument that has a bias of +3.0%. The

simulator reference value of 0.082 g/210L has a combined uncertainty (standard deviation) of 0.0014 g/210L as determined from n=10 measurements on the gas chromatograph. Assume the combined biological and analytical components of variation are found from: SD = 0.0305B+0.0026. Correct the subject's mean breath alcohol concentration and find the 99% confidence interval by combining all sources of uncertainty. Use t=2.57 and equations 3,4 and 14.

11. Assume that your Toxicology Lab receives a CRM from Cerilliant with an unbiased reference value of 0.1025 g/100ml and a combined uncertainty of 0.0012 g/100ml. The Toxicology Lab performs replicate measurements (n=5) on this CRM and obtains a maximum bias of 0.003 g/100ml and a standard deviation of 0.0007 g/100ml on a mean value of 0.0995 g/100ml. The Toxicology Lab does not correct for bias. Next, the Toxicology Lab prepares simulator solutions and performs replicate measurements (n=15) obtaining a mean of 0.1005 g/100ml with a standard deviation of 0.0006 g/100ml. When this solution is heated in a simulator to 34^0C, a breath test instrument obtained a mean value from replicate measurements (n=10) of 0.0815 g/210L with a standard deviation of 0.0010 g/210L during a calibration procedure. Assume a partition coefficient of 1.23 with an uncertainty of 0.0124. Determine the bias in the breath test instrument along with a combined uncertainty. Set up an uncertainty budget as well.

12. Assume that you are preparing an ethanol reference standard. Your preparation function is:

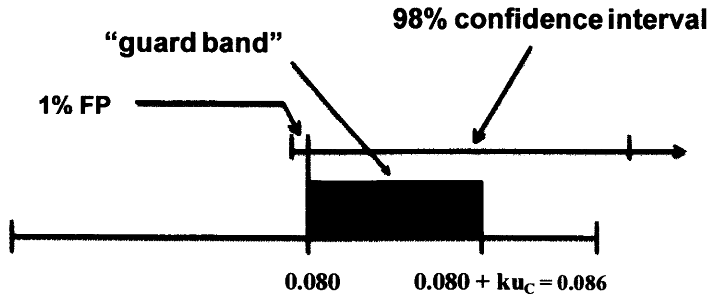
$$C = \frac{m_{Etoh} P D}{m_{Solvent}}$$

where: C = concentration of ethanol (g/100ml)
m_{Etoh} = mass measure of ethanol (g)
$m_{Solvent}$ = mass measure of the solvent (g)
P = purity as a mass fraction
D = density of the solution (g/ml)

You carefully weigh out three grams of ethanol which has a combined uncertainty (standard deviation) estimate of 0.008g. This combined uncertainty includes traceability, replication and scale resolution. The purity of the ethanol is reported as 0.992 ± 0.002. Assume the uniform distribution for this uncertainty estimate. The density of the solution is 0.65 g/ml with an uncertainty of 0.0006 g/ml. You mix the ethanol with water to a total mass for the solvent of 1.8Kg with an uncertainty of 0.009Kg. Determine the concentration C and the combined uncertainty. Set up an uncertainty budget as well.

13. In the UK a DUI case is not prosecuted until the mean BAC result is at least 0.086 g/100ml (or 86 mg/dL). What is the estimate of their combined uncertainty (u_c) if this "guard band" is ensuring a 99%

probability that the true BAC exceeds 0.080 g/100ml? Consider the following figure and solve for u_c:



14. Assume that during a one month period for a particular jurisdiction there were 458 women and 1,860 men arrested for DUI. Amongst these, 64 women and 298 men refused to submit to a breath test. Construct a 95% confidence interval for the proportion that refuse the test for each gender. Construct also a 95% confidence interval for the difference between the two proportions. Finally, construct a two-way contingency table that uses the χ^2 test to evaluate for independence between gender and refusal rate. Assume Z=1.96 for the confidence intervals. Would you conclude there is a significant difference between the two refusal rates? Use equations 17, 18 and 19.

15. Assume that an individual provides duplicate breath samples resulting in 0.092 and 0.098 g/210L. When a new simulator solution (reference value = 0.082 g/210L determined from 30 measurements with a standard deviation of 0.0010 g/210L) was installed on the instrument about two weeks earlier the first five results were: 0.086, 0.084, 0.083, 0.086 and 0.083 g/210L. Determine the 99% confidence interval for the within-subject population mean breath alcohol concentration. Include the uncertainty in the breath test instrument measurement of the simulator standard and the gas chromatography measurement of the simulator solution. Find the standard deviation for the subject's results from: SD = 0.0305B+0.0026 and use t=2.57.
Use Equations 3, 4 and 5.

16. An individual provides duplicate breath alcohol results of 0.088 and 0.095 g/210L. Assume the standard deviation associated with these individual analyses is 0.0053 g/210L. How long would you have to go back in time prior to the analyses in order to perform duplicate analyses and assume that you would be able to measure a difference in the sample means? Assume β = 0.015 g/210L/hr. and that you need a critical difference of: $\delta_{cr} = 2.77(S_{\bar{Y}})$

17. You are evaluating the accuracy of a breath test instrument and have performed n=10 simulator measurements and obtained a mean of 0.099 g/210L and a standard deviation of 0.0012 g/210L. The simulator solution you used was tested by gas chromatography and yielded a mean

of 0.103g/210L with a standard deviation of 0.001 g/210L in m=30 measurements. Construct a 99% confidence interval for the difference between the breath test instrument results and the reference value (the systematic error). Express the confidence interval as percentages also. Use Equation 7.

18. An article in the literature reported the study of acutely ill diabetic patients in a condition known as diabetic keto-acidosis (Owen, O.E., Trapp, V.E., Skutches, C.L., Mozzoli, M.A., Hoeldtke, R.D., Boden, G. and Reichard, G.A., "Acetone Metabolism During Diabetic Ketoacidosis", Diabetes, Vol.31, 1982, pp. 242-248). The maximum reported plasma acetone concentration observed was 8.91 mM (milliMoles) per liter. Assume the blood/breath partition coefficient for acetone is 300 and that 58 g of acetone equals 1 Mole. If an infrared breath test instrument requires 642 μg/L of acetone in the breath to yield 0.01 g/210L ethanol equivalent, what result would we expect from the extreme patient reported?

19. Assume that the number of tests performed per day on a particular breath test instrument follow the Poisson distribution with mean $\lambda = 40$. If 55 or more tests are performed per day then a second instrument will be necessary to install. Determine the probability that 55 or more tests will be performed on a particular day. Hint: Use the normal approximation to the Poisson where both the expected value (mean) and variance is λ.

20. A study ("Ethanol Metabolism in Men and Women", Journal of Studies on Alcohol, Vol.48, 1987, pp. 380-387) compared the elimination rates of ethanol in men and women. The study found (table 1 of the study) the mean elimination rate and for men was $\beta_M = 17.24$ mg/dl/hr (n=75) while that of women was $\beta_F = 20.77$ mg/dl/hr (m=59). The standard errors of the mean ($\boldsymbol{SE_{\bar{Y}} = S_{\bar{Y}} = \frac{S_Y}{\sqrt{n}}}$) were also noted to be $SE_M = 0.40$ and $SE_F = 0.55$ for the men and women respectively. Use a two sample t-test (two-tailed) assuming equal variances to test the hypotheses: H_0: $\beta_M = \beta_F$ H_1: $\beta_M \neq \beta_F$ at $\alpha=0.01$. Use equation 8.

21. You are interested in determining whether there is a significant difference between simultaneously collected within-subject blood and breath alcohol samples. Since we must employ the same units for both measurements, assume that the BrAC has units of g/100ml. You are only concerned if there is a systematic difference (δ) of 0.005 g/100ml. Assume the standard deviation of the differences between the paired measurements is 0.007 g/100ml. Determine the sample size you would need in order to detect a difference of 0.005 g/100ml as being significant at a level of $\boldsymbol{\alpha=0.05}$ with a power level of 0.80. Use Equation 9.

22. Some have argued that we should not report the final BrAC at the end

of exhalation but rather we should report the average value over the exhalation time. Assume an individual exhales for 10 seconds and provides a breath alcohol exhalation curve that is modeled by:

$$B_t = 0.15(1 - e^{-2t}) + 0.003t \qquad \textit{Eq. 1}$$

Find the average value for this model at t=5 seconds and again at t=10 seconds. Determine also the function that models the average value as a function of time. Is the average value a constant percentage of the function in equation 1? (You will need to integrate equation 1 over time t and then divide this by t)

23. "Alcohol free" beers claim to contain less than 0.5% alcohol by volume. You want to test this claim so you place one 12 fluid ounce "alcohol free" beer in a simulator, heat it to 34^0 C and perform five measurements with a properly calibrated breath test instrument. You obtain the following results: 0.285, 0.281, 0.278, 0.282, 0.274. Test the null hypothesis H_0: $\mu_0 < 0.5\%$ against H_1: $\mu_0 \exists 0.5\%$ at the $\alpha=0.05$ level. Use Equations 10 and 11.

24. One individual provided n=10 replicate breath samples into a breath test instrument within 12 minutes and obtained the following results: 0.079, 0.079, 0.078, 0.078, 0.078, 0.074, 0.075, 0.075, 0.075, 0.076. Calculate the mean, standard deviation and coefficient of variation for the results based on two digit truncated results and based on all three digits. Why do we obtain different results? Should two or three digits be employed? Use Equations 12 and 13.

25. An individual provides only one breath sample resulting in 0.116 g/210L and then refuses the second. Compute a 99% confidence interval for the population mean breath alcohol concentration using the following equation to estimate the standard deviation: SD = 0.0305B+0.0026. Let t=2.57. Use Equations 3.

26. The BAC Datamaster uses the following equation to compute the breath alcohol concentration from the dc voltage generated from the detector:

$$X_a = -1.3\left[ln(1 - \frac{V}{5})\right] \qquad \textit{Eq. 1}$$

The microprocessor, however, is not capable of handling the logarithmic function so an approximation is used derived from a Taylor's series expansion. Only the first three terms of the infinite series are used which is seen below.

$$f(x) = -1.3\left[-x - \frac{x^2}{2} - \frac{x^3}{3}\right] \qquad \textit{Eq. 2} \qquad \text{where: } x = v/5$$

Compare the accuracy of using the approximation in equation 2 with the direct calculation in equation 1 when the voltage (v) = 0.40. Compare the two results when the voltage is v=1.1. Which of the two methods is always lower?

27. You obtain a blood sample from a vehicular homicide case and have it analyzed twice in your toxicology laboratory with results: 0.084 and 0.086 g/100ml. The defendant gets the sample also and has it analyzed with the following results: 0.083, 0.084, 0.083, 0.083, 0.085 g/100ml. You send it off to a third lab for re-analysis. They do two runs with duplicates each and obtain: 0.084, 0.086, 0.085, 0.088 g/100ml. Assume that none of the methods employed had any systematic error. What is the estimate of the individual's "true" blood alcohol concentration?

28. Assume that you calibrate a breath test instrument with a simulator using a mercury thermometer that is reading 0.3^0 C too low. You calibrate the instrument with a simulator solution prepared to produce a vapor alcohol concentration of 0.082 g/210L, as determined by the Toxicology Laboratory. Later, an arrested subject provides breath samples resulting in 0.125 and 0.132 g/210L. What would be the subject's corrected mean BrAC? Assume that a one degree centigrade change in temperature results in a 6.5% change in headspace alcohol concentration.

29. Assume that the elimination rate of "mouth alcohol" for a person with alcohol in their system is defined according to the following differential equation:

$$\frac{dB}{dt} = -k(B - B_0)$$

where: B = breath alcohol concentration at time t
B_0 = mean end-expiratory breath alcohol concentration
t = time

Show that solving this differential equation yields the following "mouth alcohol" elimination model with three parameters: $B = C_0 e^{-kt} + B_0$
Assume that non-linear regression of "mouth alcohol" elimination data yields the following parameter estimates:
$C_0 = 0.85\,g/210L,\ k = 0.5\,min^{-1},\ B_0 = 0.150\,g/210L$. If 0.165 g/210L represents three standard deviations above the mean end-expiratory breath alcohol concentration, at what time would the "mouth alcohol" decrease to this concentration?

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Equations For Solving Problems

1. Widmark's Equation

$$N = \frac{Wr(C_t + \beta t)}{0.82(fl.oz.\ EtOH/drink)}$$

where:
- N = number of drinks
- W = body weight in ounces
- r = Widmark's ρ (rho) (L/Kg)
- C_t = BAC(Kg/L) at time t and approximated from BrAC measurement
- β = elimination rate (Kg/L/hr)
- t = time in hours from first drink
- 0.82 = density of ethanol (oz./fl.oz.)

2. Widmark's Method For Computing the Uncertainty in Widmark's Equation

$$1\sigma = \sqrt{0.015625\,N^2 + 0.050176\left[N - \frac{0.68\,C_t W}{0.82\left(fl.oz./\,drink\right)}\right]^2} \quad \textit{for men}$$

$$1\sigma = \sqrt{0.01\,N^2 + 0.021904\left[N - \frac{0.55\,C_t W}{0.82\left(fl.oz./\,drink\right)}\right]^2} \quad \textit{for women}$$

where:
- 1σ = one standard deviation
- N = number of drinks
- C_t = blood alcohol concentration in Kg/L
- W = body weight in ounces

3. Confidence Interval for a Population Mean

$$\bar{x} \pm t_{(1-\frac{\alpha}{2})(n-1)df} \frac{S}{\sqrt{n}}$$

where:
- $\bar{x}$ = sample mean
- t = from a table of t values for appropriate level of confidence

S = the standard deviation of a single value

4. Accuracy (Systematic Error or Bias)

$$\%SE = \left[\frac{\bar{x} - R}{R}\right]100$$

where: SE = systematic error or bias
$\bar{X}$ = mean results
R = reference value

5. Error Propagation: General Formula for Random Errors

$$S_q = \sqrt{\left[\frac{\partial q}{\partial x}\right]^2 S_x{}^2 + \left[\frac{\partial q}{\partial y}\right]^2 S_y{}^2}$$

where: x and y are independent and there can be as many terms as there are independent variables

6. Confidence Interval for a Population Mean

$$P\left[\bar{x} - t_{(1-\frac{\alpha}{2})(n-1)df}\frac{S}{\sqrt{n}} \le \mu \le \bar{x} + t_{(1-\frac{\alpha}{2})(n-1)df}\frac{S}{\sqrt{n}}\right] = P$$

Where: μ = true population mean

7. Confidence Interval for a Difference Between Two Sample Means

$$diff \pm t_{(1-\frac{\alpha}{2})(n_1+n_2-2)df}\, SE_{diff}$$

where:

$$diff = \bar{X}_1 - \bar{X}_2$$

$$SE_{diff} = \sqrt{\frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}\left[\frac{1}{n_1} + \frac{1}{n_2}\right]}$$

This method assumes equal variance for both variables

8. t-Test For Independent Means Assuming Equal Variances

$$t = \frac{\overline{X}_1 - \overline{X}_2}{SE_{diff}}$$

where: SE_{diff} = standard error (standard deviation) of difference computed as follows

$$SE_{diff} = \sqrt{\frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}\left[\frac{1}{n_1} + \frac{1}{n_2}\right]}$$

where: the degrees of freedom is: $df = n_1 + n_2 - 2$

For small samples, the Wilcoxon Rank Sum test is a non-parametric equivalent to the independent sample t-test

9. Calculation of Required Sample Size for Paired Data

$$n \geq \left[\frac{\sigma}{\delta}\right]^2 \left[Z_{1-\alpha/2} + Z_{1-\beta}\right]^2$$

where:
σ = standard deviation (assumed constant)
δ = critical difference
P = power of the test $(1-\beta)$
β = typically set to 0.20 (a one sided test value)
α = significance level
Z = from the standard normal distribution

10. Water/Air Partition Coefficient for Ethanol

$$K_{water/air} = 23017.268\, e^{-0.0643T}$$

where: T = temperature in celcius

11. t-Test For A Population Mean

$$t = \frac{\overline{X} - \mu}{S_X / \sqrt{n}}$$

where: μ = the population mean
S_X = the standard deviation of a single result

12. Standard Deviation

$$S = \sqrt{\frac{\sum_{i=1}^{n} \left(x_i - \overline{x}\right)^2}{n - 1}}$$

13. Coefficient of Variation

$$CV = \left[\frac{S}{\overline{x}}\right] 100$$

14. Combining Uncertainty From Two Independent Sources

$$S_T = \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$$

where: S_T = total combined uncertainty
S_1^2 = variance from one source
S_2^2 = variance from second source

15. Equation for Computing Total Body Water

$$Men: \quad TBW = 2.447 - 0.09516 Age + 0.1074 Height + 0.3362 Weight$$
$$Women: \quad TBW = -2.097 + 0.1069 Height + 0.2466 Weight$$

where: TBW is in liters
Age is in years
Height is in centimeters (2.54cm = 1 inch)
Weight is in kilograms (1Kg = 2.2 lb)

Source: Watson, P.E., Watson, I.D. and Batt, R.D., "Prediction of

blood alcohol concentrations in human subjects", *J Stud on Alcohol*, Vol.42 No.7, 1981, pp. 547-555.

16. Widmark's Equation Using Total Body Water

$$N = \frac{\frac{TBW}{0.8}(C_t + \beta t)}{0.82(fl.oz.\ EtOH/drink)}$$

where:
- N = number of drinks
- TBW = total body water in liters
- 0.8 = fraction of blood that is water (L/L)
- C_t = BAC(g/L) at time t and approximated from BrAC measurement
- β = elimination rate (g/L/hr)
- t = time in hours from first drink
- 0.82 = density of ethanol (oz./fl.oz.)
- 28.349 g = 1 ounce by weight

17. Confidence Interval for a Large (n>30) Sample Proportion

$$p \pm Z_{1-\alpha/2}\sqrt{\frac{p(1-p)}{n}}$$

where:
- p = sample proportion
- Z = from the standard normal table for a desired level of confidence
- n = sample size

18. Confidence Interval for a Difference Between Large (each n>30) Independent Sample Proportions

$$p_1 - p_2 \pm Z_{1-\alpha/2}\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

where:
- p_1 = sample 1 proportion
- p_2 = sample 2 proportion
- Z = from the standard normal table for a desired level of confidence
- n_1 = sample 1 size

n_2 = sample 2 size

19. χ^2 Analysis for Independence in a Two-way Table

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

where: O_i = the observed values in cell i
E_i = the expected values in cell i

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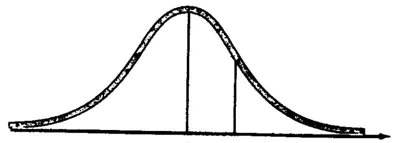
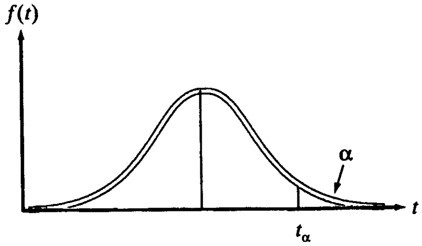
TABLE IV Normal Curve Areas



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

Source: Abridged from Table I of A. Hald, *Statistical Tables and Formulas* (New York: Wiley), 1952. Reproduced by permission of A. Hald.

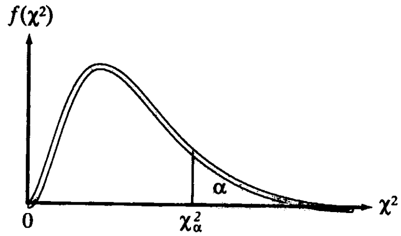
TABLE VI Critical Values of t



ν	$t_{.100}$	$t_{.050}$	$t_{.025}$	$t_{.010}$	$t_{.005}$	$t_{.001}$	$t_{.0005}$
1	3.078	6.314	12.706	31.821	63.657	318.31	636.62
2	1.886	2.920	4.303	6.965	9.925	22.326	31.598
3	1.638	2.353	3.182	4.541	5.841	10.213	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	1.319	1.714	2.069	2.500	2.807	3.485	3.767
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	1.296	1.671	2.000	2.390	2.660	3.232	3.460
120	1.289	1.658	1.980	2.358	2.617	3.160	3.373
∞	1.282	1.645	1.960	2.326	2.576	3.090	3.291

Source: This table is reproduced with the kind permission of the Trustees of Biometrika from E. S. Pearson and H. O. Hartley (eds.), *The Biometrika Tables for Statisticians*, Vol. 1, 3d ed., Biometrika, 1966.

TABLE VII Critical Values of χ^2



Degrees of Freedom	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$
1	.0000393	.0001571	.0009821	.0039321	.0157908
2	.0100251	.0201007	.0506356	.102587	.210720
3	.0717212	.114832	.215795	.351846	.584375
4	.206990	.297110	.484419	.710721	1.063623
5	.411740	.554300	.831211	1.145476	1.61031
6	.675727	.872085	1.237347	1.63539	2.20413
7	.989265	1.239043	1.68987	2.16735	2.83311
8	1.344419	1.646482	2.17973	2.73264	3.48954
9	1.734926	2.087912	2.70039	3.32511	4.16816
10	2.15585	2.55821	3.24697	3.94030	4.86518
11	2.60321	3.05347	3.81575	4.57481	5.57779
12	3.07382	3.57056	4.40379	5.22603	6.30380
13	3.56503	4.10691	5.00874	5.89186	7.04150
14	4.07468	4.66043	5.62872	6.57063	7.78953
15	4.60094	5.22935	6.26214	7.26094	8.54675
16	5.14224	5.81221	6.90766	7.96164	9.31223
17	5.69724	6.40776	7.56418	8.67176	10.0852
18	6.26481	7.01491	8.23075	9.39046	10.8649
19	6.84398	7.63273	8.90655	10.1170	11.6509
20	7.43386	8.26040	9.59083	10.8508	12.4426
21	8.03366	8.89720	10.28293	11.5913	13.2396
22	8.64272	9.54249	10.9823	12.3380	14.0415
23	9.26042	10.19567	11.6885	13.0905	14.8479
24	9.88623	10.8564	12.4011	13.8484	15.6587
25	10.5197	11.5240	13.1197	14.6114	16.4734
26	11.1603	12.1981	13.8439	15.3791	17.2919
27	11.8076	12.8786	14.5733	16.1513	18.1138
28	12.4613	13.5648	15.3079	16.9279	18.9392
29	13.1211	14.2565	16.0471	17.7083	19.7677
30	13.7867	14.9535	16.7908	18.4926	20.5992
40	20.7065	22.1643	24.4331	26.5093	29.0505
50	27.9907	29.7067	32.3574	34.7642	37.6886
60	35.5346	37.4848	40.4817	43.1879	46.4589
70	43.2752	45.4418	48.7576	51.7393	55.3290
80	51.1720	53.5400	57.1532	60.3915	64.2778
90	59.1963	61.7541	65.6466	69.1260	73.2912
100	67.3276	70.0648	74.2219	77.9295	82.3581

Source: From C. M. Thompson, "Tables of the Percentage Points of the χ^2-Distribution," *Biometrika*, 1941, 32, 188–189. Reproduced by permission of the *Biometrika* Trustees.

TABLE VII ***Continued***

Degrees of Freedom	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	2.70554	3.84146	5.02389	6.63490	7.87944
2	4.60517	5.99147	7.37776	9.21034	10.5966
3	6.25139	7.81473	9.34840	11.3449	12.8381
4	7.77944	9.48773	11.1433	13.2767	14.8602
5	9.23635	11.0705	12.8325	15.0863	16.7496
6	10.6446	12.5916	14.4494	16.8119	18.5476
7	12.0170	14.0671	16.0128	18.4753	20.2777
8	13.3616	15.5073	17.5346	20.0902	21.9550
9	14.6837	16.9190	19.0228	21.6660	23.5893
10	15.9871	18.3070	20.4831	23.2093	25.1882
11	17.2750	19.6751	21.9200	24.7250	26.7569
12	18.5494	21.0261	23.3367	26.2170	28.2995
13	19.8119	22.3621	24.7356	27.6883	29.8194
14	21.0642	23.6848	26.1190	29.1413	31.3193
15	22.3072	24.9958	27.4884	30.5779	32.8013
16	23.5418	26.2962	28.8454	31.9999	34.2672
17	24.7690	27.5871	30.1910	33.4087	35.7185
18	25.9894	28.8693	31.5264	34.8053	37.1564
19	27.2036	30.1435	32.8523	36.1908	38.5822
20	28.4120	31.4104	34.1696	37.5662	39.9968
21	29.6151	32.6705	35.4789	38.9321	41.4010
22	30.8133	33.9244	36.7807	40.2894	42.7956
23	32.0069	35.1725	38.0757	41.6384	44.1813
24	33.1963	36.4151	39.3641	42.9798	45.5585
25	34.3816	37.6525	40.6465	44.3141	46.9278
26	35.5631	38.8852	41.9232	45.6417	48.2899
27	36.7412	40.1133	43.1944	46.9630	49.6449
28	37.9159	41.3372	44.4607	48.2782	50.9933
29	39.0875	42.5569	45.7222	49.5879	52.3356
30	40.2560	43.7729	46.9792	50.8922	53.6720
40	51.8050	55.7585	59.3417	63.6907	66.7659
50	63.1671	67.5048	71.4202	76.1539	79.4900
60	74.3970	79.0819	83.2976	88.3794	91.9517
70	85.5271	90.5312	95.0231	100.425	104.215
80	96.5782	101.879	106.629	112.329	116.321
90	107.565	113.145	118.136	124.116	128.299
100	118.498	124.342	129.561	135.807	140.169