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BORKENSTEIN COURSE

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Answers to Practice Problems

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Answers to Practice Problems

1. We employ Widmark's equation as follows:

$$N = \frac{W\rho[C_t + \beta t]}{0.82(fl.oz./drink)}$$

$$N = \frac{(180lbs.)(16ozs./lb.)(0.7L/Kg)[0.00139Kg/L + (0.00017Kg/L/hr)(7hrs)]}{0.82(0.48fl.oz./drink)} = 13.2\ beers$$

13.2 beers ±25% = 13.2 ± 3.3 = 9.9 to 16.5 beers

2. We again employ Widmark's equation, this time solving for C_t as follows:

$$N = \frac{W\rho[C_t + \beta t]}{0.82(fl.oz./drink)} \rightarrow C_t = \frac{N(0.82)(fl.oz./drink)}{W\rho} - \beta t$$

$$C_t = \frac{(8\ drinks)(0.82oz./fl.oz.)(0.4fl.oz./drink)}{(125)(16)(0.60)} - (0.00015g/Kg/hr)(5) = 0.00144Kg/L = 0.144g/100ml$$

0.144 g/100ml ±40% = 0.144 ±0.058 = 0.086 to 0.202 g/100ml

3. We employ Widmark's equation but this time the number of drinks (N) is the contribution of two types of drinks. We solve as follows:

$$N_1(0.82(fl.oz./\ drink_1)) + N_2(0.82(fl.oz./\ drink_2)) = W\rho[C_t + \beta t]$$

Put in the given data and solve for C_t:

$$9(0.82(0.48)) + 5(0.82(0.4)) = (160)(16)(0.72)[C_t + (0.00018)(6)]$$

C_t = BAC = 0.173 g/100ml ±25% = 0.173 ±0.043 = 0.130 to 0.216 g/100ml

4. We begin by employing Widmark's equation as follows:

$$N = \frac{(190lbs.)(16ozs./lb.)(0.7L/Kg)[0.000885Kg/L + (0.00014Kg/L/hr)(7hrs)]}{0.82(0.48fl.oz./drink)} = 10.1\ beers$$

Assuming the ±25% uncertainty interval we obtain:

10.1 beers ±25% = 10.1 ±2.5 = 7.6 to 12.6 beers

Solving for Widmark's estimate of uncertainty we employ the following equation for men:

$$1\sigma = \sqrt{0.015625\,N^2 + 0.050176\left[N - \frac{0.68\,C_t W}{0.82\left(fl.oz./\,drink\right)}\right]^2} \quad \textit{for men}$$

Putting in our estimates we obtain:

$$1\sigma = \sqrt{0.015625\left(10.1\right)^2 + 0.050176\left[10.1 - \frac{0.68\left(0.000885\right)\left(190\right)\left(16\right)}{0.82\left(0.48\,fl.oz./\,drink\right)}\right]^2}$$

$$1\sigma = \sqrt{3.09} = 1.76\,beers$$

Based on Widmark's uncertainty estimate we would obtain:

10.1 beers ± 2(1.76) = 10.1 ± 3.52 = 6.5 to 13.6 beers

5. We first begin by employing Widmark's equation as follows:

$$N = \frac{(170lbs.)(16ozs./lb.)(0.72L/Kg)[0.001485Kg/L + (0.00018Kg/L/hr)(6hrs)]}{0.82(0.48fl.oz./drink)} = 12.7\ beers$$

Assuming the ±25% uncertainty interval we obtain:

12.7 beers ±25% = 12.7 ± 3.2 = 9.5 to 15.9 beers

We now estimate the number of drinks using the equation developed by Watson, et.al. We must convert 5 feet 10 inches into 70 inches and then into 177.8 cm given that 2.54cm = 1 inch. We must also convert 170lbs into 77.3Kg given that 1Kg = 2.2lbs. For a male the equation for TBW is:

$$TBW = 2.447 - 0.09516\left(38\,yrs\right) + 0.1074\left(177.8\,cm\right) + 0.3362\left(77.3\,Kg\right) = 43.92\,L$$

We now employ Widmark's equation based on TBW as follows:

$$N = \frac{\frac{TBW}{0.8}(C_t + \beta t)}{0.82(fl.oz.\ EtOH/drink)} = \frac{\frac{43.92L}{0.8\,L/L}\left(1.485\,g/L + \left(0.18\,g/L/h\right)\left(6h\right)\right)}{0.82\left(fl.oz.\,EtOH/\,drink\right)} = \frac{140.8\,g}{0.3936\,oz/\,drink}$$

$$N = \frac{4.97\,oz}{0.3936\,oz/\,drink} = 12.6\,beers \quad \pm 25\% \Rightarrow 9.4\ to\ 15.8\,beers$$

6. We begin by determining the systematic error associated with the test and then correct for this amount. Taking the mean of the first three simulator standards we find: mean = 0.0813 g/210L.

We then determine the systematic error according to:

$$SE = \left[\frac{\overline{X} - R}{R}\right] \bullet 100 = \left[\frac{0.0813 - 0.083}{0.083}\right] \bullet 100 = -2.05\%$$

Assuming there is no uncertainty in the estimate of the systematic error, the individual's mean breath alcohol results could then be increased by 2.05% according to:

$$\overline{X}_{Corr} = \left[\frac{\overline{X}}{1 + bias}\right] = \left[\frac{0.1015}{1 + (-0.0205)}\right] = \left[\frac{0.1015}{0.9795}\right] = 0.1036\ g/210L$$

We then employ the equation for a 99% confidence interval:

$$\overline{X}_{Corr} \pm t_{(1-\alpha/2)df=\infty}\frac{SD_i}{\sqrt{n}} = 0.1036 \pm 2.57\frac{0.0058}{\sqrt{2}} = 0.1036 \pm 0.011$$

This would yield a 99% confidence interval of: 0.0926 to 0.1146 g/210L

7. We first determine the 99% confidence interval from:

$$\overline{X}_{Corr} \pm t_{(1-\alpha/2)df=\infty}\, u_c = 0.0855 \pm 2.57\,(0.0032) = 0.0855 \pm 0.0082 \Rightarrow 0.077 \text{ to } 0.094\ g/210L$$

From this we see that the interval overlaps the critical 0.080 g/210L limit. Next, we employ the basic equation for a confidence interval which is:

$$P\left[\overline{X} - Z_{(1-\alpha/2)}\, u_{\bar{c}} \leq \mu \leq \overline{X} + Z_{(1-\alpha/2)}\, u_{\bar{c}}\right] = P$$

We begin by noticing that we are interested only in the probability that μ exceeds a lower limit and we do not care about the upper limit. So we let the upper limit go to infinity:

$$P\left[\overline{X} - Z_{(1-\alpha/2)}\, u_{\bar{c}} \leq \mu \leq \infty\right] = P$$

Next, we notice that we are interested in the probability that μ exceeds 0.080 g/210L which is the same as the value for the lower limit in the above probability equation. We set the two equal, introduce our known information and solve for $Z_{(1-\alpha/2)}$:

$$\overline{X} - Z_{(1-\alpha/2)}\, u_{\bar{c}} = 0.080 \Rightarrow 0.0855 - Z_{(1-\alpha/2)}(0.0032) = 0.080 \Rightarrow Z_{(1-\alpha/2)} = 1.72$$

We then rearrange our probability statement and introduce our determined value for $Z_{(1-\alpha/2)}$:

$$P\left[\overline{X} - Z_{1-\alpha/2}\, u_{\bar{c}} \le \mu\right] = P\left[\frac{\overline{X} - \mu}{u_{\bar{c}}} \le Z_{1-\alpha/2}\right] = P\left[Z \le Z_{1-\alpha/2}\right] = P\left[Z \le 1.72\right] = 0.9573$$

Using the standard normal tables we see that P(Z < 1.72) = 0.9573. There is a probability of 0.9573 that the individual's true mean breath

8. We begin by writing down each of our five sources of uncertainty and their estimates. They can be written down as follows:

	Traceability	Analytical	Dilutor	Bias	Total Method
Mean	0.100g/100ml	0.1025g/100ml	10.12µL	0.003g/100ml	0.0885g/100ml
SD	0.0003g/100ml	0.0011g/100ml	0.05µL		0.0013g/100ml
n		8	10		2

We now incorporate these estimates into our equation for computing combined uncertainty, assuming all components are independent and using the method of combining CV's squared as follows:

$$\frac{S_Y}{\bar{Y}} = \sqrt{CV_T^2 + CV_A^2 + CV_D^2 + CV_B^2 + CV_M^2}$$

where: CV_T^2 = uncertainty due to traceability
CV_A^2 = uncertainty due to the GC replicates
CV_D^2 = uncertainty due to the dilutor
CV_B^2 = uncertainty due to the bias
CV_M^2 = uncertainty due to the total method

$$\frac{U_{\bar{Y}}}{0.0885} = \sqrt{\left[\frac{0.0003}{0.100}\right]^2 + \left[\frac{\frac{0.0011}{\sqrt{8}}}{0.1025}\right]^2 + \left[\frac{\frac{0.05}{\sqrt{10}}}{10.12}\right]^2 + \left[\frac{\frac{0.002}{\sqrt{3}}}{0.1025}\right]^2 + \left[\frac{\frac{0.0013}{\sqrt{2}}}{0.0885}\right]^2} \Rightarrow U_{\bar{Y}} = 0.0014\,g/100ml$$

The sum of all terms under the radical is: 0.0002608 From this we find the percent contribution from each component:

Traceability:	3%
Analytical:	6%
Dilutor:	1%
Bias:	49%
Method:	41%

9. We begin by computing the mean of these results which is 0.0915 g/210L. Next, we need to estimate the standard deviation associated with the difference as follows:

$$d = Y_1 - Y_2 \quad \Rightarrow \quad V(d) = V(Y_1) + V(Y_2) \quad \Rightarrow \quad V(d) = 2V(Y_i)$$

$$S_d = \sqrt{2}\ S_{Y_i}$$

$$d \pm 1.96\,S_d \quad \Rightarrow \quad d \pm 1.96\sqrt{2}\,S_{Y_i} \quad \Rightarrow \quad 0.013 \pm 2.77(0.0054)$$

$$0.013 \pm 0.015 \qquad -0.002 \text{ to } 0.028\ g/210L$$

Since this 95% confidence interval includes zero we conclude these differences are acceptable.

For mean results at 0.250 g/210L we compute:

$$d \pm 1.96\,S_d \quad \Rightarrow \quad d \pm 1.96\sqrt{2}\,S_{Y_i} \quad \Rightarrow \quad d \pm 2.77(0.0102)$$

$$d \pm 0.028\ g/210L$$

At this concentration our differences should not exceed 0.028 g/210L.

10. We begin by finding the mean result for the subject's duplicate tests:

$$0.088/0.091\ g/210L \quad \Rightarrow \quad \overline{Y} = 0.0895\ g/210L$$

Knowing that we have a bias of +3.0%, we correct this mean result according to:

$$\overline{Y}_{Corr} = \left[\frac{\overline{Y}}{1+bias}\right] = \left[\frac{0.0895}{1+0.03}\right] \Rightarrow \overline{Y}_{Corr} = 0.0869\ g/210L$$

We now want to find the 99% confidence interval for this corrected mean. The standard error of this mean, however, is determined from two sources: the combined analytical and biological component and the reference standard. Therefore, our confidence interval estimate appears as:

$$0.0869 \pm 2.57\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$$

where: S_1^2 = the variance determined from the equation given in the problem which combines both analytical and biological components

S_2^2 = the variance representing the uncertainty in the reference value

From the equation given in the problem we estimate the standard deviation combining both analytical and biological components as:

$$S = 0.0305B + 0.0026 \;=\; 0.0305(0.0869) + 0.0026 \;=\; 0.0053\,g/210L$$

We now put our variance estimates into the confidence interval estimate:

$$0.0869 \pm 2.57\sqrt{\frac{0.0053^2}{2} + \frac{0.0014^2}{10}} \;\Rightarrow\; 0.0869 \pm 0.0097$$

This results in the confidence interval of: 0.0772 to 0.0966 g/210L We do not use a pooled estimate of the variance in this example because the two variance components are largely different. The problem can also be solved by estimating a combined uncertainty from the CV's for each contributing element. In this case we estimate the confidence interval from:

$$0.0869 \pm 2.57\,S_{\bar{Y}}$$

$$where: \frac{S_{\bar{Y}}}{0.0869} = \sqrt{CV_T^2 + CV_R^2} \;\Rightarrow\; \frac{S_{\bar{Y}}}{0.0869} = \sqrt{\left[\frac{\frac{0.0053}{\sqrt{2}}}{0.0869}\right]^2 + \left[\frac{\frac{0.0014}{\sqrt{10}}}{0.082}\right]^2} \;\Rightarrow\; S_{\bar{Y}} = 0.0038\,g/210L$$

Notice that we have solved directly for the standard deviation of the mean by incorporating the appropriate sample sizes for each component. We now find the confidence interval from:

$$0.0869 \pm 2.57(0.0038) \;\Rightarrow\; 0.0869 \pm 0.0098$$

This results in the confidence interval of: 0.0771 to 0.0967 g/210L which are almost identical to those estimated above.

11. This problem presents six sources of combined uncertainty that should be reported as part of the final breath test instrument calibration results. These four sources are: (1) the CRM traceability, (2) the Toxicology Lab bias, (3) the Toxicology Lab uncertainty in measuring the CRM, (4) the Toxicology Lab uncertainty in measuring the simulator solution, (5) the partition coefficient uncertainty and (6) the breath test instrument uncertainty. Each component was also determined based on differing number of measurements. We incorporate these different values of n into the square root sign and arrive directly at a standard deviation (uncertainty) for the mean. The value for each of these expressed as a CV are:

$$CV_{CRM} = \frac{0.0012}{0.1025} \qquad CV_{ToxBias} = \frac{\frac{0.003}{\sqrt{3}}}{0.0995} \qquad CV_{ToxCRM} = \frac{\frac{0.0007}{\sqrt{5}}}{0.0995} \qquad CV_{ToxSol} = \frac{\frac{0.0006}{\sqrt{15}}}{0.1005}$$

$$CV_{PartCoef} = \frac{0.0124}{1.23} \qquad CV_{Inst} = \frac{\frac{0.0010}{\sqrt{10}}}{0.0815}$$

Notice that we have incorporated the sample sizes only for those components that we had information for. Notice also that we have assumed a uniform distribution for the Toxicology Lab bias and estimated its uncertainty by dividing by the square root of three. The vapor alcohol reference value in the simulator is found by dividing 0.1005 by 1.23 and obtaining 0.0817 g/210L. We now simply combine these sources of uncertainty as follows:

$$\frac{u_C}{\overline{Y}} = \sqrt{CV_{CRM}^2 + CV_{ToxBias}^2 + CV_{ToxCRM}^2 + CV_{ToxSol}^2 + CV_{PartCoef}^2 + CV_{Inst}^2}$$

$$\frac{u_C}{0.0815} = \sqrt{\left[\frac{0.0012}{0.1025}\right]^2 + \left[\frac{\frac{0.003}{\sqrt{3}}}{0.0995}\right]^2 + \left[\frac{\frac{0.0007}{\sqrt{5}}}{0.0995}\right]^2 + \left[\frac{\frac{0.0006}{\sqrt{15}}}{0.1005}\right]^2 + \left[\frac{0.0124}{1.23}\right]^2 + \left[\frac{\frac{0.0010}{\sqrt{10}}}{0.0815}\right]^2}$$

$$\frac{u_C}{0.0815} = 0.0239 \quad \Rightarrow \quad u_C = 0.0019\ g/210L \quad \textit{the standard error of the mean}$$

$$bias = \left[\frac{\overline{Y} - R}{R}\right] = \left[\frac{0.0815 - 0.0817}{0.0817}\right] \bullet 100 = -0.24\%$$

From these results we can set up an uncertainty budget and list the percent contribution from each factor:

Factor	**Uncertainty**	**Type**	**Percent**
CRM	0.0012	B	24%
Tox Lab Bias	0.0017	A	53%
Tox Lab CRM	0.0003	A	1%
Tox Lab Sol	0.00016	A	1%
Part Coef	0.0124	B	18%
Inst	0.0003	A	3%

12. We first need to determine the uncertainty (standard deviation) estimates for each of the three input variables. They are as follows:

	Estimate	Uncertainty
Mass of ethanol:	3.00 g	0.008 g
Purity:	0.992	$u_P = \frac{0.002}{\sqrt{3}} = 0.00116$
Density:	0.65 g/ml	0.0006 g/ml
Mass of solvent:	1.8 Kg	0.009 Kg

We first solve for the concentration C according to:

$$C = \frac{m_{Etoh}\,P\,D}{m_{Solvent}} = \frac{(3\,g)(0.992)(0.65\,g/ml)}{1800\,g} = 0.00107\,g/ml$$

We now combine these four components using the method of general error propagation and assuming independence of all components:

$$u_C = \sqrt{\left[\frac{\partial C}{\partial m_{Etoh}}\right]^2 S^2_{m_{Etoh}} + \left[\frac{\partial C}{\partial P}\right]^2 S^2_P + \left[\frac{\partial C}{\partial D}\right]^2 S^2_D + \left[\frac{\partial C}{\partial m_{Solvent}}\right]^2 S^2_{m_{Solvent}}}$$

$$u_C = \sqrt{\left[\frac{PD}{m_{Solvent}}\right]^2 0.008^2 + \left[\frac{m_{Etoh}\,D}{m_{Solution}}\right]^2 0.00116^2 + \left[\frac{m_{Etoh}P}{m_{Solvent}}\right]^2 0.0006^2 + \left[-\frac{m_{Etoh}PD}{m^2_{Solvent}}\right]^2 9^2}$$

$$u_C = \sqrt{[0.000358]^2\,0.008^2 + [0.00108]^2\,0.00116^2 + [0.00165]^2\,0.0006^2 + [-0.0000006]^2\,9^2}$$

$$u_C = \sqrt{0.00000000004} = 0.0000063\,g/ml = 0.00063\,g/100ml$$

Solving for the uncertainty by the method of summing the CV squared yields:

$$\frac{u_C}{C} = \sqrt{CV^2_{m_{Etoh}} + CV^2_P + CV^2_D + CV^2_{m_{Solvent}}} \Rightarrow \frac{u_C}{0.00107} = \sqrt{\left[\frac{0.008}{3}\right]^2 + \left[\frac{\frac{0.002}{\sqrt{3}}}{0.992}\right]^2 + \left[\frac{0.0006}{0.65}\right]^2 + \left[\frac{9}{1800}\right]^2}$$

$$\Rightarrow \frac{u_C}{0.00107} = 0.00586$$

$$u_C = 0.0000063\,g/L = 0.00063\,g/100ml \quad \Rightarrow \quad CV = 0.59\%$$

These results are the same because the preparation function for the ethanol concentration is multiplicative.

From these results we can set up an uncertainty budget and list the percent contribution from each factor:

Factor	**Uncertainty**	**Type**	**Percent**
Mass of ethanol	0.008	A	21%
Purity	0.0012	B	4%
Density	0.0006	B	2%
Mass of solvent	9.0	A	73%

13. Here we are interested in estimating the combined uncertainty, u_c, from the equation: $0.080 + k\,u_c = 0.086$ First we must look up the value of k from the standard normal table. Since we want only a 1% chance that the true value could be below 0.080 g/100ml, we look up the value for which there is 99% area to the left, based on how the table is constructed. From the table this gives us the value of k=2.33. Solving then for u_c we obtain:

$$0.080 + 2.33\,u_c = 0.086 \Rightarrow 2.33\,u_c = 0.006 \Rightarrow u_c = 0.0026\,g/100ml$$

14. We first compute the confidence intervals for each gender as follows:

For women:

$$p \pm Z_{1-\alpha/2}\sqrt{\frac{p(1-p)}{n}} \Rightarrow \frac{64}{458} = 0.14 \pm 1.96\sqrt{\frac{0.14(0.86)}{458}} \Rightarrow 0.14 \pm 0.032 \Rightarrow 0.108\,to\,0.172$$

For men:

$$p \pm Z_{1-\alpha/2}\sqrt{\frac{p(1-p)}{n}} \Rightarrow \frac{298}{1860} = 0.16 \pm 1.96\sqrt{\frac{0.16(0.84)}{1860}} \Rightarrow 0.16 \pm 0.017 \Rightarrow 0.143\,to\,0.177$$

From this we see a slight overlap between the two confidence intervals

Next, we find the confidence interval for the difference between the two proportions according to:

$$p_1 - p_2 \pm Z_{1-\alpha/2}\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}} \Rightarrow 0.14 - 0.16 \pm 1.96\sqrt{\frac{0.14(0.86)}{458} + \frac{0.16(0.84)}{1860}}$$
$$\Rightarrow -0.02 \pm 0.036 \Rightarrow -0.056 \text{ to } 0.016$$

From this we see that the 95% confidence interval for the difference includes zero. Thus we conclude there is not significant difference.

We now set up the two-way contingency table and enter the observed and expected values as follows:

	Provide Test	Refused Test		Proportion Refusing
Women	394 386.5	64 71.5	458	14.0%
Men	1562 1569.5	298 290.5	1860	16.0%
	1956	362	2,318	

Our test hypotheses are: H_0 *: gender and refusal rates are independent*
H_A *: gender and refusal rates are not independent*

From the table values we compute the χ^2 statistic as follows:

$$\chi^2=\sum_{i=1}^{k}\frac{(O_i-E_i)^2}{E_i}=\frac{(394-386.5)^2}{386.5}+\frac{(64-71.5)^2}{71.5}+\frac{(1562-1569.5)^2}{1569.5}+\frac{(298-290.5)^2}{290.5}=1.17$$

For a two-way table the degrees of freedom are: $df=(rows-1)(columns-1)=1$
The critical χ^2 statistic for df=1 and α=0.05 is found from the table to be: $\chi^2_{0.95,df=1}=3.841$. Since our computed χ^2 statistic is much less than this table value, we do not reject the null hypothesis that there is no association between gender and refusal rate. This further supports the confidence interval for differences between the two proportions.

15. This problem illustrates the more complete analysis for determining the uncertainty in forensic breath alcohol results. This will incorporate the uncertainty from the three major sources: the subject, the breath test instrument and the gas chromatograph used to measure the simulator solution standard. We will begin by identifying the following variables: Y = subject BrAC results, X = simulator results, R = reference value determined from gas chromatograph, Z = subject BrAC results after correcting for any systematic error and S = standard deviation estimates. We now find the mean and standard deviation of the subject's results: mean = 0.095 g/210L and from the equation for the standard deviation we obtain:

S = 0.0305(0.095) + 0.0026 = 0.0055 g/210L

The standard deviation for the mean is found from:

$$S_{\overline{Y}} = \frac{S_Y}{\sqrt{n}} = \frac{0.0055}{\sqrt{2}} = 0.0039\, g / 210L$$

Next, given the mean reference standard result of 0.082 g/210L and the standard deviation of 0.0010 g/210L with n=30 measurements, we find the standard deviation of the mean according to:

$$S_{\overline{R}} = \frac{S_R}{\sqrt{n}} = \frac{0.0010}{\sqrt{30}} = 0.00018\, g / 210L$$

Next, we find the mean and standard deviation resulting from the instrument's measurements of the first five simulator standards:

mean = 0.0844 g/210L and S = 0.0015 g/210L from which we find that the standard deviation of the mean is: $S_{\overline{X}} = \frac{S_X}{\sqrt{n}} = \frac{0.0015}{\sqrt{5}} = 0.00067\, g / 210L$

The confidence interval we will employ is: $\overline{Z} \pm t_{(1-\alpha/2)df=\infty} \frac{S_Z}{\sqrt{n}}$

where $\overline{Z}$ is the mean BrAC result corrected for any systematic error. Since $\frac{S_Z}{\sqrt{n}}$ is the standard deviation of the mean we will simply re-express this as: $S_{\overline{Z}}$. Our corrected confidence interval expression will be: $\overline{Z} \pm t_{(1-\alpha/2)df=\infty} S_{\overline{Z}}$. Based on the information given in the problem we find the corrected BrAC according to: $\overline{Z} = \left[\frac{\overline{Y}}{1+bias}\right] = \frac{\overline{Y}R}{\overline{X}}$

where: the bias = the systematic error. The bias is found to be:

$bias = \left[\frac{\overline{X}-R}{R}\right] = \left[\frac{0.0844-0.082}{0.082}\right] = +0.029$ We now find our corrected mean to be:

$$\overline{Z} = \left[\frac{\overline{Y}}{1+0.029}\right] = 0.095\, [0.9718] = 0.0923\, g / 210L$$

Next, we express more fully the equation for the corrected mean BrAC:

$$\overline{Z} = \left[\frac{\overline{Y}}{1+\frac{\overline{X}-R}{R}}\right] = \frac{\overline{Y}R}{\overline{X}}$$

Based on this equation, we see that $\overline{Z}$ is a function of three measured variables: the mean of the subject's results, the mean of the instrument measurement of the simulator standards and the mean of the gas chromatograph measurements of the simulator standard. All three have uncertainty that must be included in the uncertainty estimate for $\overline{Z}$, $S_{\overline{Z}}$. To determine this we use equation 5 from the attached page and assume that all three sources of uncertainty are independent:

$$S_{\overline{Z}} = \sqrt{\left[\frac{\partial \overline{Z}}{\partial \overline{Y}}\right]^2 S_{\overline{Y}}^2 + \left[\frac{\partial \overline{Z}}{\partial R}\right]^2 S_R^2 + \left[\frac{\partial \overline{Z}}{\partial \overline{X}}\right]^2 S_{\overline{X}}^2}$$

Introducing our data into this equation we obtain:

$$S_{\overline{Z}} = \sqrt{\left[\frac{0.082}{0.0844}\right]^2 \left[\frac{0.0055}{\sqrt{2}}\right]^2 + \left[\frac{0.095}{0.0844}\right]^2 \left[\frac{0.0010}{\sqrt{30}}\right]^2 + \left[-\frac{(0.095)(0.082)}{(0.0844)^2}\right]^2 \left[\frac{0.00152}{\sqrt{5}}\right]^2}$$

This results in: $S_{\overline{Z}} = \sqrt{0.00001487} = 0.00386\ g/210L$

Another approach to finding the standard deviation estimate for $\overline{Z}$ is to employ coefficient of variations since the model is multiplicative. Some find this approach easier. We write the general uncertainty equation in the form:

$$CV_{\overline{Z}} = \frac{S_{\overline{Z}}}{\overline{Z}} = \sqrt{CV_{\overline{Y}}^2 + CV_R^2 + CV_{\overline{X}}^2}$$

We now incorporate our estimates into this equation for the CV's:

$$\frac{S_{\overline{Z}}}{0.0922} = \sqrt{\left[\frac{\frac{0.0055}{\sqrt{2}}}{0.095}\right]^2 + \left[\frac{\frac{0.0010}{\sqrt{30}}}{0.082}\right]^2 + \left[\frac{\frac{0.0015}{\sqrt{5}}}{0.0844}\right]^2} = 0.04174$$

$$S_{\overline{Z}} = (0.0922)(0.04174) = 0.00385\ g/210L$$

Now we compute our corrected confidence interval:

$$\overline{Z} \pm t_{(1-\alpha/2)df=\infty} S_{\overline{Z}} \Rightarrow 0.0922 \pm 2.57(0.0039) \Rightarrow 0.0822\ to\ 0.1022$$

Our confidence interval now includes uncertainty from both random and systematic sources and is therefore a more complete analysis.

16. In this problem we want to determine the time necessary for the individual to eliminate sufficient alcohol so that the mean of duplicate measurements can be sufficiently distinguished. We begin by

solving for the critical difference necessary for two means to have (given the measurement variability) in order to be able to conclude that they will be measured as different:

$$\delta_{cr} = 2.77\,SD_{\bar{x}} = 2.77\left[\frac{0.0053}{\sqrt{2}}\right] = 0.010\ g/210L$$

We now put this critical difference value into an equation that will solve for the time necessary to wait: $\mathbf{0.010 = 0.015t \Rightarrow t = 0.66\ hrs.}$

17. We are interested here in determining the 99% confidence interval for a population parameter δ, the true difference between the mean of the instrument results and the mean of the reference standard determinations. The relationship is as follows:

$$\delta = \mu_X - \mu_R \Rightarrow \hat{\delta} = \hat{\mu}_X - \hat{\mu}_R \Rightarrow \hat{\delta} = \overline{X} - \overline{R}$$

Our 99% confidence interval for δ which assumes equal variance for both X and R is:

$$\hat{\delta} \pm t_{1-\alpha/2}\, SE_{\hat{\delta}} = \hat{\delta} \pm t_{1-\alpha/2}\sqrt{\frac{(n-1)S_X^2 + (m-1)S_R^2}{n+m-2}\left[\frac{1}{n} + \frac{1}{m}\right]}$$

Putting our data into this equation yields:

$$-0.004 \pm 2.712\sqrt{\frac{(9)(0.0012)^2 + (29)(0.001)^2}{38}\left[\frac{1}{10} + \frac{1}{30}\right]} = -0.004 \pm 0.0010$$

The 99% confidence interval for δ would be: -0.005 to -0.003 g/210L. In percentage of systematic error this corresponds to:

$$\left[\frac{-0.005}{0.103}\right]100 = -4.85\% \qquad \left[\frac{-0.003}{0.103}\right]100 = -2.9\%$$

-4.85% to -2.9%

18. Much of the biomedical literature reports concentrations of substances in many different units (i.e., as mM or milliMoles in this case). Since mM is actually an expression of mM/L, we must first convert the mM to grams. We do this by observing that 1 Mole of acetone weighs 58 grams. Therefore we obtain:

$$8.91\,mM = 0.00891\,M \quad \Rightarrow \quad \frac{X}{58\,g} = \frac{0.00891\,M}{1\,M} \quad \Rightarrow \quad X = 0.517\,g/L$$

Assuming the value of $K_{bl/br}$ = 300 we obtain:

$$300 = \frac{0.517\,g/L}{Y} \quad \Rightarrow \quad Y = 0.00172\,g/L = 1720\,\mu g/L \;\; \textit{in the breath}$$

If 642 μg/L are necessary to yield 0.01 g/210L ethanol equivalent then:

$$\frac{1720\,\mu g/L}{642\,\mu g/L} = \frac{Z}{0.01\,g/210L} \quad \Rightarrow \quad Z = 0.027\,g/210L$$

19. We begin by identifying the following variable which has the Poisson distribution:

X = the number of tests performed on the instrument per day

Since X can be viewed as the sum of several other Poisson random variables (e.g., the number of tests per minute or per hour) and $\lambda = 40$ is fairly large (e.g., > 20) we will assume that X has a normal distribution by the Central Limit Theorem. Since we are using a continuous distribution (the normal) to approximate a discrete random variable (X) we will use a continuity correction of ½. We incorporate our data and solve as follows:

$$P[X \geq 55] = P\left[\frac{X - E(X)}{\sqrt{V(X)}} > \frac{(55 - \frac{1}{2}) - E(X)}{\sqrt{V(X)}}\right] = P\left[Z > \frac{54.5 - 40}{\sqrt{40}}\right] = P[Z > 2.29] = 0.011$$

We obtain the probability of 0.011 from the standard normal tables. This is a very small probability which suggests that it is unlikely that 55 or more tests will be performed on the instrument in one day.

20. Here we have two random variables of interest: X_F = the alcohol elimination rate for women and X_M = the alcohol elimination rate for men. Since we do not know what their exact distributions are we do know that by the Central Limit Theorem their means will be approximately normally distributed for sample sizes greater than approximately 30. We write each of their distributions showing the expected values (means) and variances as follows:

$$X_F \sim (\beta_F, \sigma_F^2) \Rightarrow \overline{X}_F \sim N(\beta_F, \frac{\sigma_F^2}{n}) \qquad X_M \sim (\beta_M, \sigma_M^2) \Rightarrow \overline{X}_M \sim N(\beta_M, \frac{\sigma_M^2}{n})$$

We now identify the hypotheses that we want to test:

H_0: $\beta_F = \beta_M$ $\qquad\qquad$ H_1: $\beta_F \neq \beta_M$

We use the two sample t-test since we do not know σ_F^2 or σ_M^2 exactly and have to use our sample variances S_F^2 and S_M^2 as estimates. We will assumed equal variances. We use equation 8:

$$t = \frac{\overline{X}_F - \overline{X}_M}{\sqrt{\frac{(n_1 - 1)S_F^2 + (n_2 - 1)S_M^2}{n_1 + n_2 - 2} \bullet \left[\frac{1}{n_1} + \frac{1}{n_2}\right]}}$$

Since the study presented the standard errors for each mean we need to determine the standard deviations as follows:

$$SE_{\overline{X}_F} = \frac{SD}{\sqrt{n}} \Rightarrow SD = 0.55(\sqrt{59}) = 4.22 \qquad SE_{\overline{X}_M} = \frac{SD}{\sqrt{n}} \_ \ SD = 0.40(\sqrt{75}) = 3.46$$

We then introduce our information into the equation for t as follows:

$$t = \frac{20.77 - 17.24}{\sqrt{\frac{(59 - 1)(17.8) + (75 - 1)(12.0)}{59 + 75 - 2} \bullet \left[\frac{1}{59} + \frac{1}{75}\right]}} = 5.34$$

Our degrees of freedom for this problems are n+m-2 = 59+75-2 = 132. From the t-table we find $t_{0.995,132} = 2.576$. Our calculated t value is 5.34 which is even larger than 2.576. The probability of obtaining a value as large as 5.34 when the null hypothesis is true is even smaller than 0.005. Therefore, we conclude that there is a significant difference between the elimination rates of men and women in this study.

21. We use equation 9. We note that α=0.05 for the two-tailed test and so we look up the value for $Z_{0.975}$ in the standard normal table. This value is 1.96. We use this two tailed value because we did not specify that our critical difference of 0.005 g/100ml was only in one direction. Next we look up the $Z_{0.8}$ in the standard normal table which is 0.85. We then put these values along with the information given into equation 9 as follows:

$$n \geq \left[\frac{0.007\ g/100ml}{0.005\ g/100ml}\right]^2 [1.96 + 0.85]^2 = 15.5 \approx 16$$

So, we would need 16 individuals for our study.

22. The average value of a smooth continuous function of time is determined by integrating the given function over the time interval involved and then dividing by the time interval. The average value over the time interval of 5 seconds is determined by:

$$\int_0^t B_0(1 - e^{-kt}) + Ct\,dt = \int_0^t 0.15(1 - e^{-2t}) + 0.003t\,dt = 0.15t + \frac{0.15}{2}e^{-2t} + \frac{0.003\,t^2}{2}\Big|_0^5 = 0.7125$$

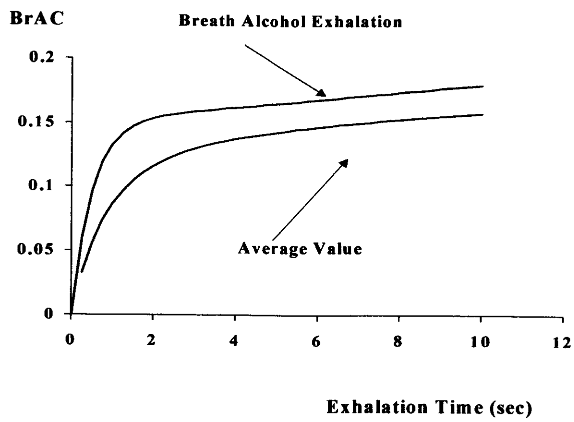
Note that the middle term involving e^{-2t} goes to zero when t=5 so it can be disregarded. Now divide this integral by the time interval of 5 seconds:

$$\frac{0.7125}{5} = 0.1425\ g/210L$$

When the same procedure is done for the exhalation time of 10 seconds we obtain an average value of 0.1575 g/210L. Comparing these average values to the actual BrAC values from equation 1 in the problem and determining the percent differences we obtain:

$$\left[\frac{0.143 - 0.165}{0.165}\right]_{t=5} \bullet 100 = -13.6\% \quad - \quad \left[\frac{0.158 - 0.18}{0.18}\right]_{t=10} \bullet 100 = -12.5\%$$

So, no, it is not a constant percentage. This problem is illustrated in the figure below:



23. First we must determine the percent by volume of alcohol associated with each of the five measurements. We first use equation 10 to determine the partition coefficient for ethanol in water heated to 34^0C in the simulator. Placing T=34 into equation 10 we obtain:

$$K_{w/a} = 23017.268\, e^{-0.0643(34)} = 2586$$

We use this result to next determine the concentration of the alcohol in the beer using the following equation:

$$K_{w/a} = \frac{C_w}{C_a} \Rightarrow 2586 = \frac{C_w}{0.285\ g/210L} \Rightarrow C_w = 0.351 g/100ml$$

We then need to determine what volume 0.351 g/100ml of ethanol occupies by using the density equation as follows:

$$D = \frac{g}{ml} \Rightarrow 0.789 = \frac{0.351\ g/100ml}{X\ ml/100ml} \Rightarrow X = 0.445 ml/100ml$$

Our result is 0.445 ml/100ml which is equivalent to 0.445% by volume. Doing this for the remaining four measurements we obtain the following percent by volume estimates: 0.439, 0.434, 0.440 and 0.428. Their mean is: 0.4372 with a standard deviation of 0.0065. We use the following t-test since we do not know the exact population standard deviation (σ) and must estimate it with the sample standard deviation (S):

$$t_{(1-\alpha/2)df=4} = \frac{\overline{X} - \mu}{S/\sqrt{n}} = \frac{0.4372 - 0.50}{0.0065/\sqrt{5}} = -21.7$$

From the table of critical t values we observe that for 4 degrees of freedom (n-1) a t value of -4.604 would put an area in the lower tail of 0.005. Our t result of -21.7 is much lower than the table value. This means the area to the left of -21.7 is much smaller than 0.005. This is interpreted to mean that the probability of obtaining the results we did if the null hypothesis were true is much less than 0.005. We therefore reject the null hypothesis and conclude that there is evidence in support of the alternate hypothesis, that the population mean alcohol concentration for this "alcohol free" beer is less than 0.5% by volume.

24. Using the truncated two digit and three digit results we obtain the following:

Two digit: mean = 0.070 g/210L sd = 0 CV = 0%
Three digit: mean = 0.0767 g/210L sd = 0.0019 g/210L CV = 2.45%

The significant differences are due to using the different number of digits. One should clearly use the three digit data in this example.

25. We would compute the confidence interval in the same manner as we did in problem #5. We assume the one result is a random sample from a population having the same standard deviation as we would estimate if they had provided two samples. The one sample was still received from the same instrument method and general protocol. We now use the following equation for the 99% confidence interval:

$$\overline{X} \pm t_{(1-\alpha/2)df=\infty} \frac{SD_i}{\sqrt{n}} = 0.116 \pm 2.57 \frac{0.0061}{\sqrt{1}} = 0.116 \pm 0.016$$

This would yield a 99% confidence interval of: 0.100 to 0.132 g/210L

26. First computing the breath alcohol from equation 1 we obtain:

$$X_a = -1.3\left[\ln(1 - \frac{V}{5})\right] = X_a = -1.3\left[\ln(1 - \frac{0.08}{5})\right] = 0.108396\ g/210L$$

Next, computing the result from the approximation we obtain:

$$f(x) = -1.3(-x - \frac{x^2}{2} - \frac{x^3}{3}) = -1.3(-0.08 - \frac{0.08^2}{2} - \frac{0.08^3}{3}) = 0.108382 \text{ g/210L}$$

Computing the results with a v=1.1 we obtain: 0.322999 g/210L from the direct method and 0.322075 from the approximation method. At all concentrations the approximation method used by the instrument will be less than that found by direct calculation using equation 1. Since the approximation is an expansion about v=0, the approximation result will deviate further from the direct calculation as results increase above zero. The deviation, however, is always to become lower than the direct calculation. Moreover, this illustrates that using three terms in the expanded series is more than enough for three digit final results.

27. We have three different estimates (three labs) of this person's BAC where each has different n and different variance (S_i^2) estimates. We will want to use a weighted mean in which the weights will be determined according to n and the variance for each lab.

$$\bar{Y}_w = \frac{\sum_{i=1}^{n} w_i \bar{Y}_i}{\sum_{i=1}^{n} w_i} \qquad \textit{where}: \; w_i = \frac{n_i}{S_i^2}$$

where: w_i = the weight for the i^{th} lab

$$\bar{Y}_w = \frac{\left[\frac{2}{0.000002}\right](0.0850) + \left[\frac{5}{0.0000008}\right](0.0836) + \left[\frac{4}{0.0000029}\right](0.0858)}{\left[\frac{2}{0.000002}\right] + \left[\frac{5}{0.0000008}\right] + \left[\frac{4}{0.0000029}\right]} = 0.0839 \text{ g/100ml}$$

28. If the thermometer is reading 0.3^0 C too low, then the simulator headspace actually contained more alcohol than was thought. Therefore, a higher concentration than realized was introduced into the instrument during calibration. This would result in the instrument measuring the alcohol concentration systematically low. Given that a one degree centigrade change results in a 6.5% change in headspace alcohol concentration, we can determine the percent change resulting from a 0.3^0 degree centigrade change according to:

$$\frac{0.065}{1^0\,C} = \frac{X}{0.3^0\,C} \Rightarrow X = 0.0195 \Rightarrow 1.95\%$$

Based on this result, the instrument is reading all results 1.95% systematically low. Therefore, the subject's mean BrAC should be adjusted up by 1.95%. This is done by:

$$0.125 / 0.132 \Rightarrow \bar{X} = 0.1285 \quad \Rightarrow 0.1285(1.0195) = 0.131 \, g / 210L$$

29. We solve the differential equation by integrating after the separation of variables:

$$\frac{dB}{dt}=-k(B-B_0) \Rightarrow \frac{dB}{B-B_0}=-k\,dt \Rightarrow \int\frac{dB}{B-B_0}=\int -k\,dt \Rightarrow ln\,B-B_0=-kt+C$$

Now exponentiating both sides we obtain:

$$B-B_0=e^{-kt+C} \Rightarrow B-B_0=C_0\,e^{-kt} \Rightarrow B=C_0e^{-kt}+B_0$$

Given the initial conditions of:

$$B(0)=C_{max} \Rightarrow C_{max}=C_0e^{-k(0)}+B_0 \Rightarrow C_{max}=C_0+B_0 \Rightarrow C_0=C_{max}-B_0$$

From the non-linear regression we obtained the three parameter estimates which we now put into our model:

$$B=0.85e^{-0.5t}+0.150$$

We solve for t when B = 0.165 g/210L:

$$0.165=0.85e^{-0.5t}+0.150 \Rightarrow 0.015=0.85e^{-0.5t} \Rightarrow 0.0176=e^{-0.5t}$$

Taking the natural log of both sides we obtain:

$$ln\,0.0176=ln\,e^{-0.5t} \Rightarrow -4.04=-0.5t \Rightarrow t=8.1\,min\,utes$$

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